| Student Name: ____________________ | Contact Person Name: ____________________ |
|---|---|
| Date: ____________________ | Phone Number: ____________________ |

Texas Assessment of Knowledge and Skills

# Exit Level Math Review

# Lesson 19

## Distance, midpoint, and 3D figures

**TAKS Objective 7 –** Demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes

**Lesson Objectives:**

- Use the Pythagorean theorem to derive the distance and midpoint formulas
- Use the distance and midpoint formulas
- Analyze characteristics of common 3D figures

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The Texas Assessment of Knowledge and Skills (TAKS) exit level exam covers ten learning objectives. These lessons are designed to teach math concepts specific to each objective as well as strategies to consider when approaching typical TAKS questions. To successfully complete the TAKS exit level exam, the student should be able to:

1) **Describe functional relationships in a variety of ways.**
2) **Demonstrate an understanding of the properties and attributes of functions.**
3) **Demonstrate an understanding of linear functions.**
4) **Formulate and use linear equations and inequalities.**
5) **Demonstrate an understanding of quadratic equations and other nonlinear functions.**
6) **Demonstrate an understanding of geometric relationships and spatial reasoning.**
7) **Demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.**
8) **Demonstrate an understanding of concepts and uses of measurement and similarity.**
9) **Demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.**
10) **Demonstrate an understanding of the mathematical processes and tools used in problem solving.**


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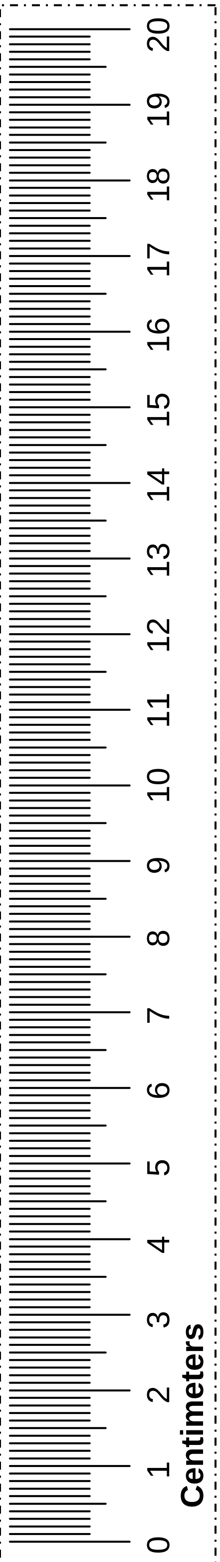
# *TAKS* Mathematics Chart




## Length

| Metric | Customary |
|---|---|
| 1 kilometer = 1000 meters | 1 mile = 1760 yards |
| 1 meter = 100 centimeters | 1 mile = 5280 feet |
| 1 centimeter = 10 millimeters | 1 yard = 3 feet |
| | 1 foot = 12 inches |

## Capacity and Volume

| Metric | Customary |
|---|---|
| 1 liter = 1000 milliliters | 1 gallon = 4 quarts |
| | 1 gallon = 128 fluid ounces |
| | 1 quart = 2 pints |
| | 1 pint = 2 cups |
| | 1 cup = 8 fluid ounces |

## Mass and Weight

| Metric | Customary |
|---|---|
| 1 kilogram = 1000 grams | 1 ton = 2000 pounds |
| 1 gram = 1000 milligrams | 1 pound = 16 ounces |

## Time

1 year = 365 days

1 year = 12 months

1 year = 52 weeks

1 week = 7 days

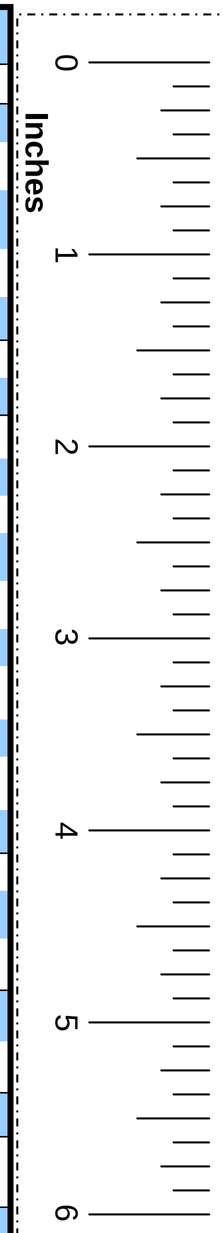
1 day = 24 hours

1 hour = 60 minutes

1 minute = 60 seconds

# *TAKS* Mathematics Chart

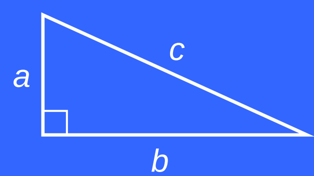
| | | |
|---|---|---|
| **Perimeter** | Rectangle | $P = 2l + 2w$ or $P = 2(l + w)$ |
| **Circumference** | Circle | $C = 2\pi r$ or $C = \pi d$ |
| **Area** | Rectangle | $A = lw$ or $A = bh$ |
| | Triangle | $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$ |
| | Trapezoid | $A = \frac{1}{2}(b_1 + b_2)h$ or $A = \frac{(b_1 + b_2)h}{2}$ |
| | Regular polygon | $A = \frac{1}{2}aP$ |
| | Circle | $A = \pi r^2$ |
| ***P*** represents the perimeter of the base of a three-dimensional figure. | | |
| ***B*** represents the area of the base of a three-dimensional figure. | | |
| **Surface Area** | Cube (total) | $S = 6s^2$ |
| | Prism (lateral) | $S = Ph$ |
| | Prism (total) | $S = Ph + 2B$ |
| | Pyramid (lateral) | $S = \frac{1}{2}Pl$ |
| | Pyramid (total) | $S = \frac{1}{2}Pl + B$ |
| | Cylinder (lateral) | $S = 2\pi rh$ |
| | Cylinder (total) | $S = 2\pi rh + 2\pi r^2$ or $S = 2\pi r(h + r)$ |
| | Cone (lateral) | $S = \pi rl$ |
| | Cone (total) | $S = \pi rl + \pi r^2$ or $S = \pi r(l + r)$ |
| | Sphere | $S = 4\pi r^2$ |
| **Volume** | Prism or Cylinder | $V = Bh$ |
| | Pyramid or Cone | $V = \frac{1}{3}Bh$ |
| | Sphere | $V = \frac{4}{3}\pi r^3$ |
| **Special Right** | 30°, 60°, 90° | $x,\ x\sqrt{3},\ 2x$ |
| **Triangles** | 45°, 45°, 90° | $x,\ x,\ x\sqrt{2}$ |
| **Pythagorean Theorem** | | $a^2 + b^2 = c^2$ |
| **Distance Formula** | | $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ |
| **Slope of a Line** | | $m = \frac{y_2 - y_1}{x_2 - x_1}$ |
| **Midpoint Formula** | | $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ |
| **Quadratic Formula** | | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| **Slope-Intercept Form of an Equation** | | $y = mx + b$ |
| **Point-Slope Form of an Equation** | | $y - y_1 = m(x - x_1)$ |
| **Standard Form of an Equation** | | $Ax + By = C$ |
| **Simple Interest Formula** | | $I = prt$ |



***Think Back***

***The Pythagorean Theorem relates the lengths of the legs to the length of the hypotenuse of a right triangle.***

$$a^2 + b^2 = c^2$$



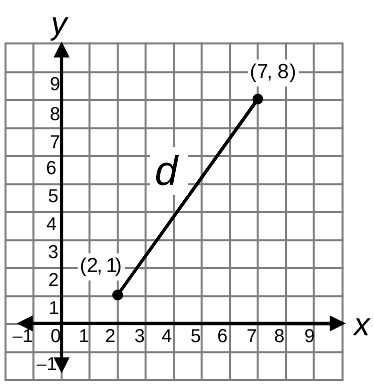

The Pythagorean Theorem has been proven in over 150 different ways. One other formula provided on your formula sheet is a direct consequence of the Pythagorean Theorem – the distance formula. This lesson, we will use the Pythagorean Theorem to show why the distance and midpoint formulas are true. We will then take a brief look at figures in three dimensions.

# The Distance formula

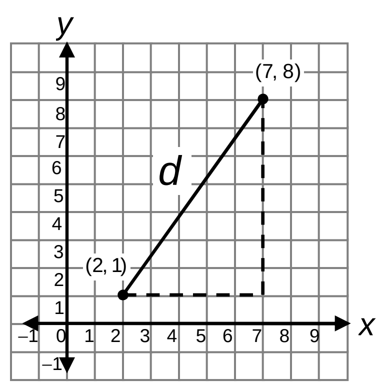
***Example***

Find the distance between the points (2,1) and (7,8).

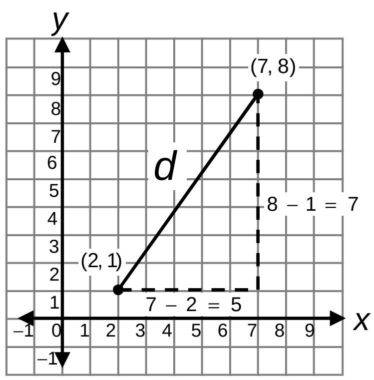
***Solution***




To begin, graph and connect these two points. Label the line segment *d*. Its length is what we wish to find.




Next, sketch a right triangle whose hypotenuse is *d*.




Find the length of each leg by counting boxes or by subtraction.
(big coordinate − small coordinate)

Finally, substitute these leg lengths into the Pythagorean Theorem and solve for *d*.

$$a^2 + b^2 = c^2$$

$$5^2 + 7^2 = d^2$$

To get rid of the "squared," we take the square root of both sides.

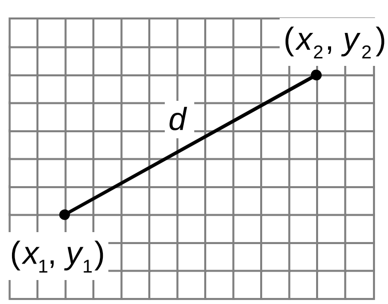
$$\sqrt{5^2 + 7^2} = \sqrt{d^2}$$

This cancels the "squared."

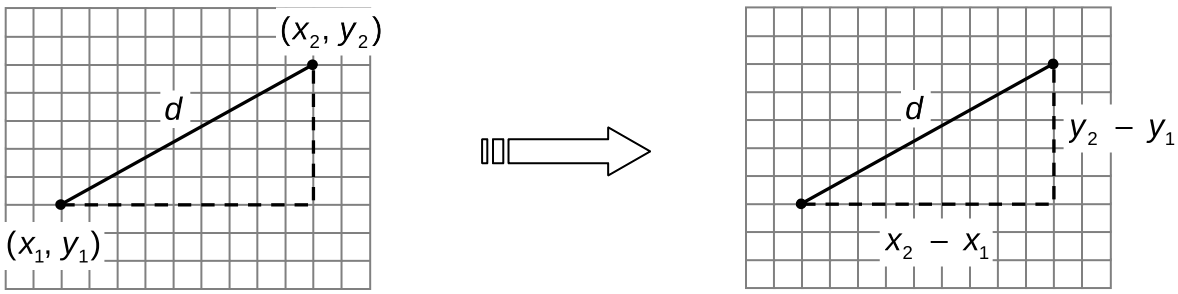
$$\sqrt{5^2 + 7^2} = d$$

$$d = \sqrt{5^2 + 7^2}$$

We did not simplify further so that you can better understand the true proof of the distance formula. To find the distance between any two points, we will use general coordinates $(x_1, y_1)$ and $(x_2, y_2)$.




Create a right triangle using *d* as its hypotenuse (shown on the next page.)




Using the coordinates, we find that the length of the base is $x_2 - x_1$; the length of the vertical leg is $y_2 - y_1$.

Next, we will substitute the length of both legs into the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

Finally, solve this equation for *d* by taking the square root of both sides.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{d^2}$$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The result is the distance formula.

## Problem Solving Tip

Some students benefit from knowing why a formula is true. If this proof confuses you, do not panic. You are given the distance formula on the test. You will not need to prove it.

***Example***

What is the length of a line segment with endpoints (3,8) and (7, 2), rounded to the nearest tenth?

***Solution***

Use the distance formula.

To help substitute, begin by writing $(x_1, y_1)$ above the first given endpoint and $(x_2, y_2)$ above the second.

| |
|---|
| ***Example*** |
| $(x_1, y_1)$ $(x_2, y_2)$ |
| What is the length of a line segment with endpoints (3,8) and (7, 2)? |

Next, copy the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(7 - 3)^2 + (2 - 8)^2}$$

Substitute corresponding values.

$$d = \sqrt{(4)^2 + (-6)^2}$$

Begin simplifying from the parentheses outward.

$$d = \sqrt{16 + 36}$$

$$d = \sqrt{52}$$

$$d = 7.211...$$

The end result of the distance formula should always be <u>positive</u>.

$$d \approx 7.2$$

***Example***

What is the length of a line segment with endpoints (-4,-1) and (3,-8) rounded to the nearest tenth?

***Solution***

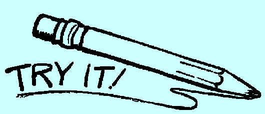
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(3 - -4)^2 + (-8 - -1)^2}$$

$$d = \sqrt{(7)^2 + (-7)^2}$$

$$d = \sqrt{49 + 49}$$

$$d = \sqrt{98} \approx 9.9$$




**1)** What is the length of a line segment with endpoints (4,3) and (7,1) rounded to the nearest tenth?

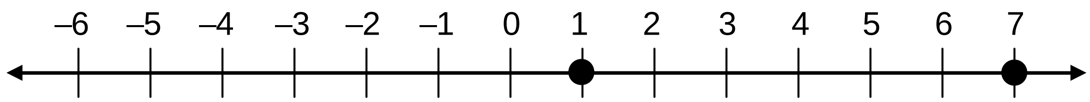
**2)** What is the length of a line segment with endpoints (-9,9) and (-2,-1) rounded to the nearest tenth?

# The Midpoint formula

To understand the midpoint formula, we will begin work with only one dimension: a number line.

***Example***

Find the midpoint of 1 and 7.




***Solution***

Find the difference between 1 and 7.

$7 - 1 = 6$

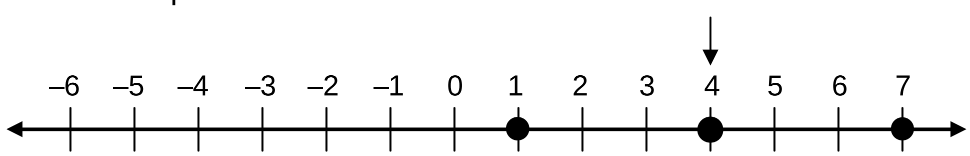
Find half of that distance.

$\frac{6}{2} = 3$

Add this to the smaller value given.

$3 + 1 = 4$

4 is the midpoint of 1 and 7.




This process is how the midpoint formula is derived.


Start with two points, $x_1$ and $x_2$. Since they are not equal, we can say $x_1 < x_2$.

The distance between them is $x_2 - x_1$.

Half this distance is $\frac{x_2 - x_1}{2}$.

Adding the smaller value to this distance, we get $\frac{x_2 - x_1}{2} + x_1$.

Finally, we simplify this and get the midpoint formula for one dimension.

$$\frac{x_2 - x_1}{2} + x_1$$

$$= \frac{x_2}{2} - \frac{x_1}{2} + x_1$$

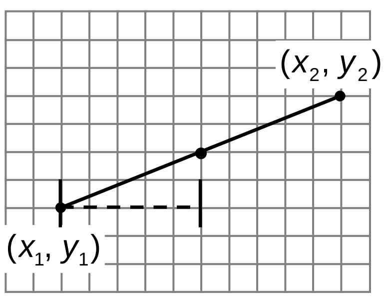
$$= \frac{x_2}{2} - \frac{x_1}{2} + \frac{2x_1}{2}$$

$$= \frac{x_2}{2} + \frac{x_1}{2}$$

$$= \frac{x_2 + x_1}{2} = \frac{x_1 + x_2}{2}$$

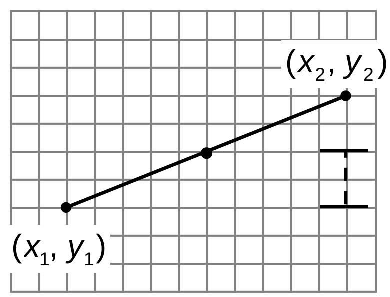
We can extend this process into two dimensions.

The basic idea is that to find the midpoint of two points, we only want to travel halfway across and halfway up or down from the first point to the second.




The $x$-coordinate of the midpoint is half the horizontal distance.

$$\frac{x_1 + x_2}{2}$$




The $y$-coordinate of the midpoint is half the vertical distance.

$$\frac{y_1 + y_2}{2}$$

Therefore, the midpoint formula is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Again, the proof is only provided to help you understand the formula. You are only required to be able to use the formula.

***Example***

Find the midpoint of the line segment with endpoints (-2,-4) and (3,7).

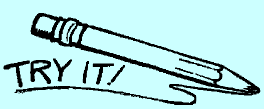
***Solution***

We will use the formula.

$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$M = \left( \frac{-2 + 3}{2}, \frac{-4 + 7}{2} \right)$$

$$M = \left( \frac{1}{2}, \frac{3}{2} \right)$$




**3)** Find the midpoint of (4,8) and (0,-4).

***Example***

$\Delta ABC$ has vertices at $A(0, 0)$, $B(9, 12)$, and $C(25, 0)$. What is the distance between the midpoint of $\overline{AB}$ and the midpoint of $\overline{AC}$?

**A** 7.5 units

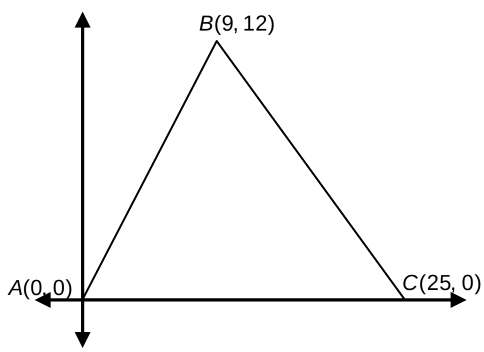
**B** 10 units

**C** 15 units

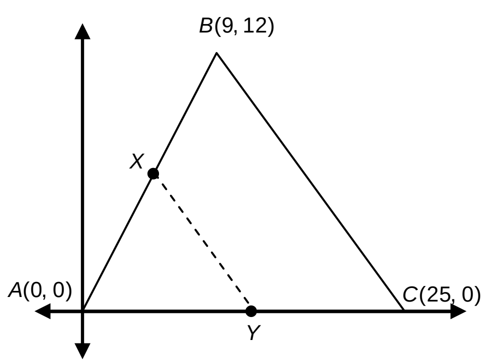
**D** 20 units

***Solution***

If you notice, the coordinates given do not make graphing very easy to do. Instead, we will sketch the relative position of the given points, so that we can better visualize the problem.




Sketch the triangle.




Next, sketch in the midpoint of $\overline{AB}$ and the midpoint of $\overline{AC}$. We will call them $X$ and $Y$. Connect them. This is the distance we wish to find.

Now we will solve the problem using formulas.

*Step 1*: Find the coordinates of $X$ and $Y$ using the midpoint formula.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$X = \left(\frac{0 + 9}{2}, \frac{0 + 12}{2}\right) \qquad Y = \left(\frac{0 + 25}{2}, \frac{0 + 0}{2}\right)$$

$$\underset{(x_1,\, y_1)}{X = (4.5, 6)} \qquad \underset{(x_2,\, y_2)}{Y = (12.5, 0)}$$

*Step 2*: Use the distance formula to find $XY$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(12.5 - 4.5)^2 + (0 - 6)^2}$$

$$d = \sqrt{(8)^2 + (-6)^2}$$

$$d = \sqrt{64 + 36}$$

$$d = \sqrt{100} = 10$$

The answer is choice **B**.

***Example***

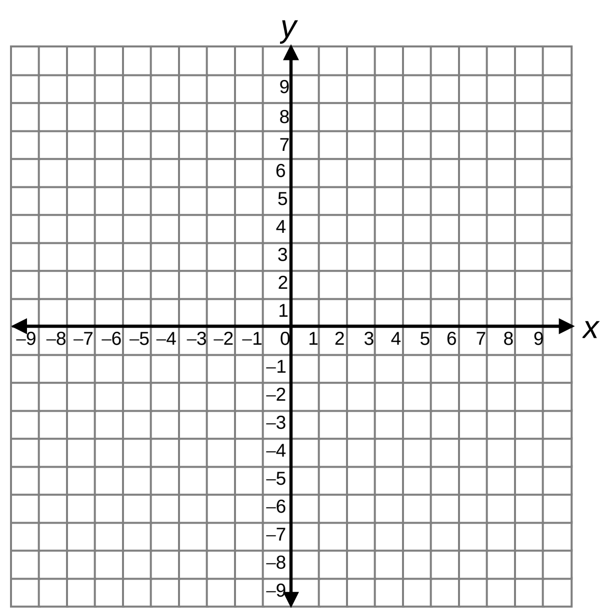
A coordinate grid is placed over a map. City A is located at $(-4, 3)$ and city B is located at $(3, 9)$. If City C is at the midpoint between City A and City B, which is closest to the distance in coordinate units from City A to City C?

**A** 4.61 units

**B** 6.52 units

**C** 9.22 units

**D** 21.25 units



***Solution***

This question asks you to find the distance between City A and City C, where City C is the midpoint of City A and B. City C is halfway between City A and B. Therefore, if we divide the distance between City A and City B by two, this will give the distance between A and C.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(3 - -4)^2 + (9 - 3)^2}$$

$$d = \sqrt{(7)^2 + (6)^2}$$

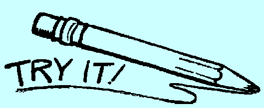
$$d = \sqrt{49 + 36}$$

$$d = \sqrt{85} = 9.2195445$$

Careful, this is not the answer.

$$\frac{d}{2} = \frac{9.2195445}{2} \approx 4.6$$

The answer is choice **A.**




**4)** Circle $Q$ has diameter $\overline{WY}$. Point $W$ is located at (3,-2), and point $Y$ is located at (5,-6). Which of the following ordered pairs represents point $Q$, the center of the circle?

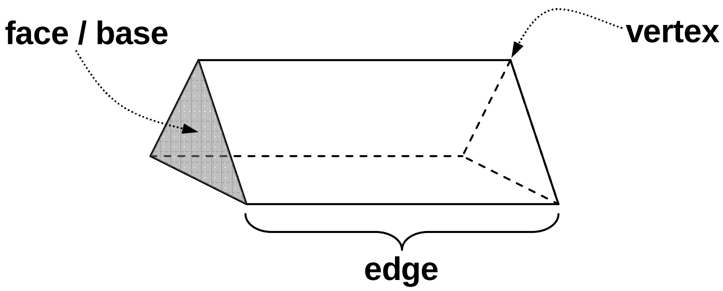
**A** (8,-8) **B** (4,-4)

**C** (-1.5,1.5) **D** (3,-6)

Hint: diameter

We will now briefly discuss the properties of certain 3-D figures.

## Triangular prism

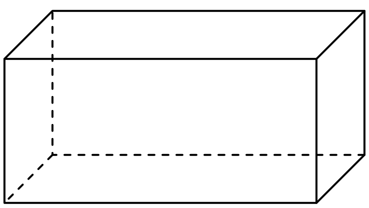



- 5 total faces
    - 2 triangular bases
    - 3 rectangular faces
- 9 edges
- 6 vertices

**FACT**

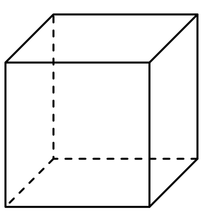
*A **face** is a flat polygon surface. An **edge** is a line segment where faces meet. A **vertex** is a point where three or more edges meet.*

## Rectangular prism



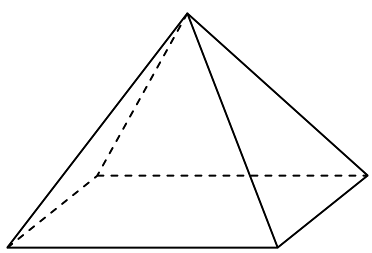
- 6 total faces
    - 2 rectangular bases
    - 4 rectangular faces
- 12 edges
- 8 vertices

## Cube



- 6 total faces
    - 2 square bases
    - 4 square faces
- 12 edges
- 8 vertices

## Square pyramid

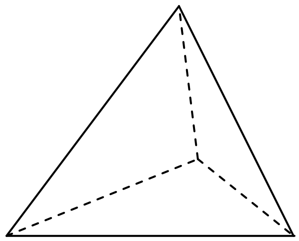


- 5 total faces
    - 1 square base
    - 4 triangular faces
- 8 edges
- 5 vertices

**FACT**

*A **B**ase is always a **F**ace, but not every **F**ace qualifies as a **B**ase. You must know the difference.*

## Triangular pyramid

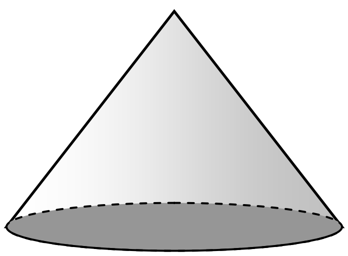


- 4 total faces
  - 1 triangular base
  - 3 triangular faces
- 6 edges
- 4 vertices

## Cylinder

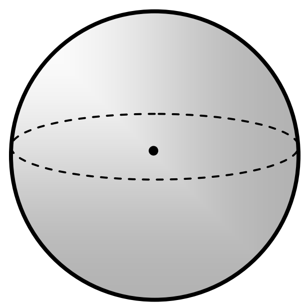
- 2 circular bases
- 1 curved surface

## Cone



- 1 circular base
- 1 curved surface
- 1 vertex

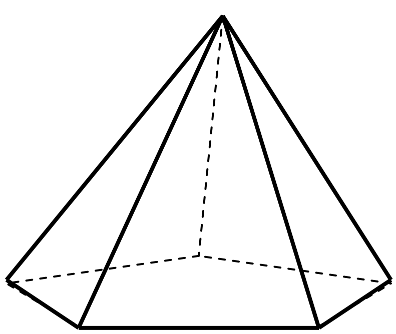
## Sphere



- 1 curved surface

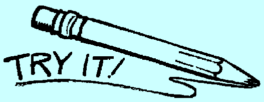
***Example***

How many faces, edges, and vertices does the pentagonal pyramid have?

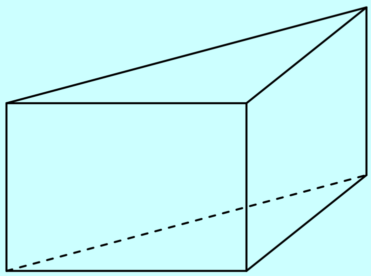


***Solution***

5 faces, 10 edges, 6 vertices.




**5)** How many faces, edges, and vertices are in the figure below?



## Review

**Know these concepts:**

**1.** The distance formula is used to find the distance between two points.

**a.** The distance between two points, $(x_1, y_1)$ and $(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**2.** Use the midpoint formula to find the location of the point equidistant from two points.

**a.** The midpoint of $(x_1, y_1)$ and $(x_2, y_2)$ is $M = \left(\frac{x_1 + x_2}{y_1 + y_2}\right)$.

**3.** Find the number of faces, edges, and vertices in a 3-dimensional figure.

# Practice Problems
## Lesson 19

Directions: Write your answers in your math journal. Label this exercise TAKS Review – Lesson 19.

**1)** Points *R*, *S*, and *T* are collinear (on the same line), and point *S* is between points *R* and *T*. The coordinate for point *R* is 20. If $RT = 24$ units and $ST = 2RS$ units, what is a coordinate for point *S*?

**A** 44 **B** 28

**C** 36 **D** 22

**2)** $\Delta XYZ$ has vertices at points $X(0, 0)$, $Y(0, 32)$, and $Z(14, 13)$. Which is closest to the distance between the midpoint of $\overline{XY}$ and $\overline{YZ}$?

**A** 9.5 units **B** 9.6 units

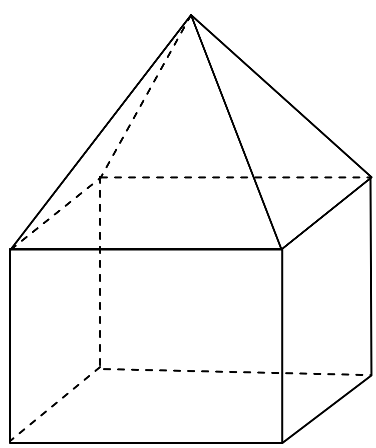
**C** (7,22.5) **D** (0,16)

**3)** Which is closest to the length of the radius of a circle with a diameter whose endpoints are $(-6, 2)$ and $(-2, 4)$?

**A** 2.24 units **B** 4.47 units

**C** (-2,3) **D** 3 units

**4)** How many faces, edges, and vertices does the solid shown have?



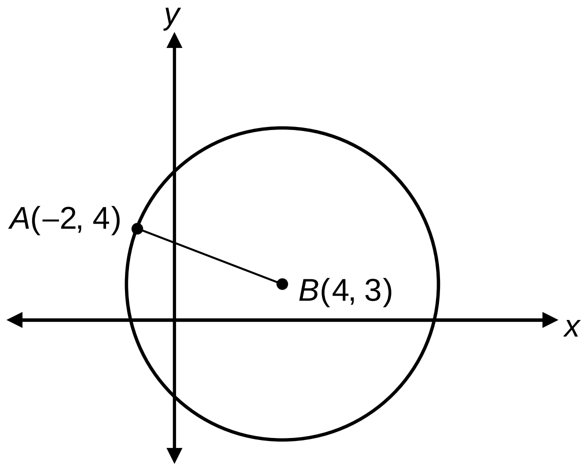
**A** 4 faces, 10 edges, and 7 vertices

**B** 9 faces, 10 edges, and 8 vertices

**C** 10 faces, 16 edges, and 9 vertices

**D** 9 faces, 16 edges, and 9 vertices

**5)** $\overline{AB}$ is the radius of the circle shown below. Which is closest to the length of the diameter of the circle?




**A** 3.1 units

**B** 4.3 units

**C** 6.1 units

**D** 12.2 units

ANSWERS TO TRY IT

**1)** $d \approx 3.6$ **2)** $d \approx 12.2$ **3)** $M = (2, 2)$

**4)** **B** **5)** 5 faces, 6 vertices, 9 edges

**End of Lesson 19**